

# Supersymmetry and the relationship between a class of singular potentials in arbitrary dimensions

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June 27, 2001

## Abstract

The eigenvalues of the potentials  $V_1(r) = \frac{A_1}{r} + \frac{A_2}{r^2} + \frac{A_3}{r^3} + \frac{A_4}{r^4}$ ,  $V_2(r) = B_1r^2 + \frac{B_2}{r^2} + \frac{B_3}{r^4} + \frac{B_4}{r^6}$ , and of the special cases of these potentials such as the Kratzer and Goldman-Krivchenkov potentials, are obtained in  $N$ -dimensional space. The explicit dependence of these potentials in higher-dimensional space is discussed, which have not been previously linked.

**Pacs Numbers:** 03.65.Fd, 03.65.Ge

## 1 Introduction

Singular potentials have attracted much attention in recent years for a variety of reasons, two of them being that (i) the ordinary perturbation theory fails badly for such potentials, and (ii) in physics, one often encounters phenomenological potentials that are strongly singular at the origin such as certain type of nucleon-nucleon potentials, singular models of fields in zero dimensions, etc. Thus a study of such potentials is of interest, both from the fundamental and applied point of view.

One of the challenging problems in non-relativistic quantum mechanics is to find exact solutions to the Schrödinger equation for potentials that can be used in different field of physics. Recently, several authors obtained exact solutions for the fourth-order inverse-power potential

$$V_1(r) = \frac{A_1}{r} + \frac{A_2}{r^2} + \frac{A_3}{r^3} + \frac{A_4}{r^4} \quad (1)$$

using analytical methods [1-3]. These methods yield exact solutions for a single state only for a potential of type (1) with restrictions on the coupling constants. The interest is mainly due to the wide applicability of these type inverse-power potentials. Some areas of interest are ion-atom scattering [4], several interactions between the atoms [5], low-energy physics [6], interatomic interactions in molecular physics [7] and solid-state physics [8].

The advent of supersymmetry has had a significant impact on theoretical physics in a number of distinct disciplines. One subfield that has been receiving much attention is supersymmetric quantum mechanics [9] in which the Hamiltonians of distinct systems are related

by a supersymmetry algebra. In this work, we are concerned with, via supersymmetric quantum mechanics, clarifying the relationship between two distinct systems having an interaction potential of type (1) and interacting through

$$V_2(r) = B_1 r^2 + \frac{B_2}{r^2} + \frac{B_3}{r^4} + \frac{B_4}{r^6} \quad (2)$$

singular even-power potentials which have been widely used in a variety of fields, e.g. see [6,10]. In recent years, the higher order anharmonic potentials have drawn more attentions of physicists and mathematicians in order to partly understand a newly discovered phenomena such as the structural phase transitions [11], the polaron formation in solids [12], the concept of false vacuo in field theory [13], fibre optics [14], and molecular physics [15]. In addition, some 60 years ago Michels *et al.* [16] proposed the idea of simulating the effect of pressure on an atom by enclosing it in an impenetrable spherical box. Since that time there have been a large number of publications, for an overview see [17], dealing with studies on quantum systems enclosed in boxes, which involve an interaction potential that is a special case ( $B_2 = 0$ ) of (2). This field has received added impetus in recent years because of the fabrication of semiconductor quantum dots [18].

The main motivation behind this work is to reveal the existence of a link between potentials of type (1) and (2) in  $N$ -dimensional space, and between their special cases such as a Mie-type potential (or Kratzer) [19] and pseudoharmonic-like (or Goldman-Krivchenkov) potential [20] in higher dimensions, which to our knowledge has never been appeared in the literature. On the other hand, with the advent of growth technique for the realization of the semiconductor quantum wells, the quantum mechanics of low-dimensional systems has become a major research field. The work presented in this letter would also be helpful to the literature in this respect as the results can readily be extended to lower dimensions as well.

## 2 The Schrödinger equation in $N$ -dimensional space

It is well known that the general framework of the non-relativistic quantum mechanics is by now well understood and its predictions have been carefully proved against observations. Physics is permanently developing in a tight interplay with mathematics. It is of importance to know therefore whether some familiar problems are a particular case of a more general scheme or to search if a map between the radial equations of two different systems exists. It is hence worthwhile to study the Schrödinger equation in the arbitrary dimensional spaces which has attracted much more attention to many authors. Many efforts have in particular been produced in the literature over several decades to study the stationary Schrödinger equation in various dimensions with a central potential containing negative powers of the radial coordinates [21, and the references therein].

The radial Schrödinger equation for a spherically symmetric potential in  $N$ -dimensional space (we shall use through this paper the natural units such that  $\hbar = m = 1$ )

$$-\frac{1}{2} \left[ \frac{d^2 R}{dr^2} + \frac{N-1}{r} \frac{dR}{dr} \right] + \frac{\ell(\ell + N - 2)}{2r^2} R = [E - V(r)] R \quad (3)$$

is transformed to

$$-\frac{d^2 \Psi}{dr^2} + \left[ \frac{(M-1)(M-3)}{4r^2} + 2V(r) \right] \Psi = 2E\Psi \quad (4)$$

where  $\Psi$ , the reduced radial wave function, is defined by

$$\Psi(r) = r^{(N-1)/2} R(r) \quad (5)$$

and

$$M = N + 2\ell \quad (6)$$

Eq.(4) can also be written as

$$-\frac{1}{2} \frac{d^2 \Psi}{dr^2} + \left[ \frac{\Lambda(\Lambda+1)}{2r^2} + V(r) \right] \Psi = E \Psi \quad (7)$$

where  $\Lambda = (M-3)/2$ . We see that the radial Schrödinger equation in  $N$ -dimensions has the same form as the three-dimensional one. Consequently, given that the potential has the same form in any dimension, the solution in three dimensions can be used to obtain the solution in any dimension simply by using the substitution  $\ell \rightarrow \Lambda$ . It should be noted that  $N$  and  $\ell$  enter into expressions (4) and (7) in the form of the combinations  $N + 2\ell$ . Hence, the solutions for a particular central potential  $V(r)$  are the same as long as  $M(= N + 2\ell)$  remains unaltered. Therefore the s-wave eigensolutions ( $\Psi_{\ell=0}$ ) and eigenvalues in four-dimensional space are identical to the p-wave solutions ( $\Psi_{\ell=1}$ ) in two-dimensions.

The technique of changing the independent coordinate has always been useful tool in the solution of the Schrödinger equation. For instance, this allows something of a systematic approach enabling to recognize the equivalence of superficially unrelated quantum mechanical problems. Many recent papers have addressed this old subject. In the light of these works we proceed by substituting  $r = \alpha\rho^2/2$  and  $R = F(\rho)/\rho^\lambda$ ,  $\lambda$  an integer, suggested by the known transformations between Coulomb and harmonic oscillator problems [22] and used to show the relation between the perturbed Coulomb problem and the sextic anharmonic oscillator in arbitrary dimensions [23,24], we transform Eq. (3) to another Schrödinger-like equation in  $N' = 2N - 2 - 2\lambda$  dimensional space with angular momentum  $L = 2\ell + \lambda$ ,

$$-\frac{1}{2} \left[ \frac{d^2 F}{d\rho^2} + \frac{N' - 1}{\rho} \frac{dF}{d\rho} \right] + \frac{L(L + N' - 2)}{2\rho^2} F = [\hat{E} - \hat{V}(\rho)] F \quad (8)$$

where

$$\hat{E} - \hat{V}(\rho) = E\alpha^2\rho^2 - \alpha^2\rho^2 V(\alpha\rho^2/2) \quad (9)$$

and  $\alpha$  is a parameter to be adjusted properly. Note that leaving re-scaling constant  $\alpha$  arbitrary for now gives us an additional degree of freedom. When we discuss bound state eigenvalues later, we can use this to allow the values of the potential coefficients to be completely independent of each other. Thus, by this transformation, in general, the  $N$ -dimensional radial wave Schrödinger equation with angular momentum  $\ell$  can be transformed to a  $N' = 2N - 2 - 2\lambda$  dimensional equation with angular momentum  $L = 2\ell + \lambda$ . If we choose  $\alpha^2 = 1/|E|$ , with  $E$  corresponding the eigenvalue for the inverse power potential of Eq. (1), then Eq. (8) corresponds to the Schrödinger equation of an singular even-power potential

$$\hat{V}(\rho) = \rho^2 + \frac{4A_2}{\rho^2} + \frac{8A_3}{\rho^4} |E|^{1/2} + \frac{16A_4}{\rho^6} |E| \quad (10)$$

with eigenvalue

$$\hat{E} = \frac{-2A_1}{|E|^{1/2}} \quad (11)$$

Thus, the system given by Eq. (1) in  $N$ -dimensional space is reduced to another system defined by Eq. (2) in  $N' = 2N - 2 - 2\lambda$  dimensional space. In other words, by changing the independent variable in the radial Schrödinger equation, we have been able to demonstrate a close equivalence between singular potentials of type (1) and (2). Note that when  $N = 3$  and  $\lambda = 0$  one finds  $N' = 4$ , and when  $\lambda = 1$  we get  $N' = 2$ . It is also easy to see that  $N' + 2L$  does not depend on  $\lambda$ , which leads to map two distinct problems in three- and four-dimensional space [24].

### 3 Mappings between two distinct systems

#### 3.1 Quasi-exactly solvable case

Since Eq. (4) for the reduced radial wave  $\Psi(r)$  in the  $N$ -dimensional space has the structure of the one-dimensional Schrödinger equation for a spherically symmetric potential  $V(r)$ , we may define the supersymmetric partner potentials [9]

$$V_{\pm}(r) = W^2(r) \pm W'(r) \quad (12)$$

which has a zero-energy solution, and the corresponding eigenfunction is given by

$$\Psi_{n=0}(r) \propto \exp[\pm \int^r W(r) dr] \quad (13)$$

In constructing these potentials one should be careful about the behaviour of the wave function  $\Psi(r)$  near  $r = 0$  and  $r \rightarrow \infty$ . It may be mentioned that  $\Psi(r)$  behaves like  $r^{(M-1)/2}$  near  $r = 0$  and it should be normalizable. For the inverse power potential of Eq. (1) we set

$$W(r) = \frac{-a}{r^2} + \frac{c}{r} - b \quad , \quad b, c > 0 \quad (14)$$

and identify  $V_+(r)$  with the effective potential so that

$$V_+(r) = \left( \frac{2A_4}{r^4} + \frac{2A_3}{r^3} + \frac{2A_2}{r^2} + \frac{2A_1}{r} \right) + \frac{(M-1)(M-3)}{4r^2} - 2E_{n=0} \quad (15)$$

and substituting Eq. (14) into Eq. (12) we obtain

$$V_+(r) = \frac{a^2}{r^4} + \frac{2a(1-c)}{r^3} + \frac{c(c-1) + 2ab}{r^2} - \frac{2bc}{r} + b^2 \quad (16)$$

and the relations between the parameters satisfy the supersymmetric constraints

$$a = \pm \sqrt{2A_4} \quad ; \quad c = 1 - \frac{A_3}{\pm \sqrt{2A_4}} \quad (17)$$

The potential (1) admits the exact solutions

$$\Psi_{n=0}(r) = N_0 r^c \exp\left(\frac{a}{r} - br\right) \quad (18)$$

where  $N_0$  is the normalization constant, with the physically acceptable eigenvalues

$$E_{n=0} = -\frac{b^2}{2} = -\frac{1}{16A_4} \left[ \frac{A_3}{\sqrt{2A_4}} \left( 1 + \frac{A_3}{\sqrt{2A_4}} \right) - \frac{1}{4} (M-1)(M-3) - 2A_2 \right]^2 \quad (19)$$

in the case of  $a < 0$  and under the constraints

$$A_1 = -(1 + \frac{A_3}{\sqrt{2A_4}})\sqrt{-2E_{n=0}} \quad (20)$$

The results obtained agree with those in Refs. [2,3,21] for three-dimensions. Note that in order to retain the well-behaved solution at  $r \rightarrow 0$  and at  $r \rightarrow \infty$  we have chosen  $a = -\sqrt{2A_4}$ .

The expressions obtained above can easily be extended to the lower dimensions. For example, one can readily check that our two-dimensional solutions ( $N = 2, \ell \rightarrow \ell - 1/2$ ) for the inverse power potential considered are in excellent agreement with the literature [21]. The ground state solutions in arbitrary dimensions for the Coulomb ( $A_2 = A_3 = A_4 = 0$ ), and for a the Kratzer ( $A_3 = A_4 = 0$ ) [19], and for an inverse-power  $A_3 = 0$  [1,2] potentials can also be found from the above prescriptions.

For the singular even-power anharmonic oscillator potential of Eq. (2), we set

$$W(r) = \mu r + \frac{\delta}{r} + \frac{\eta}{r^3} \quad , \quad \delta > 0 \quad (21)$$

which leads to

$$\Psi_{n=0}(r) = C_0 r^\delta \exp(\frac{\mu r^2}{2} - \frac{\eta}{2r^2}) \quad (22)$$

with  $C_0$  being the corresponding normalization constant, and identify  $V_+(r)$  with the effective potential so that

$$\begin{aligned} V_+(r) &= \left( \frac{2B_4}{r^6} + \frac{2B_3}{r^4} + \frac{2B_2}{r^2} + 2B_1 r^2 \right) + \frac{(M-1)(M-3)}{4r^2} - 2\tilde{E}_{n=0} \\ &= W^2(r) + W'(r) = \frac{\eta^2}{r^6} + \frac{\eta(2\delta-3)}{r^4} + \frac{\delta(\delta-1) + 2\eta\mu}{r^2} + \mu^2 r^2 + \mu(2\delta+1) \end{aligned} \quad (23)$$

and the relations between the potential parameters satisfy the supersymmetric constraints

$$\eta = \pm\sqrt{2B_4} \quad ; \quad \delta = \frac{3}{2} + \frac{B_3}{\eta} \quad ; \quad \mu = \pm\sqrt{2B_1} \quad (24)$$

As we are dealing with a confined particle system, the negative values for  $\eta$  and  $\mu$  would of course be the right choice to ensure the well behaved nature of the wave function behaviour at the origin and at infinity. Hence, physically meaningful ground state energy eigenvalues for the potential of interest are

$$\tilde{E}_{n=0} = -\frac{\mu}{2}(2\delta+1) = \sqrt{\frac{B_1}{2}} \left\{ 2 + \sqrt{1 - 16\sqrt{B_1 B_4} + 8B_2 + (M-1)(M-3)} \right\} \quad (25)$$

At this point we should report that our results reproduce those obtained by [17,25,26] when potential (2) (in case  $B_2 = 0$ ) is confined to an impenetrable spherical box in 2- and 3-dimensions. It is also not difficult to see that if one takes  $\eta = 0$  in Eq. (23), then Eq. (25) becomes the exact energy spectra of  $N$ -dimensional harmonic oscillator. Further, one easily check that in case  $B_4 = B_3 = 0$ , the above energy expression correctly reproduce the eigenvalues of the pseudo-type potential in 3-dimension [27] which is the subject of the next section.

Finally, we wish to discuss briefly the explicit mapping between the singular potentials given by Eqs. (1) and (2). If one consider the transformed anharmonic oscillator potential of Eq.

(10) and repeat the above mathematical procedure carried out through Eqs. (21-25), then the corresponding eigenvalue equation reads

$$\hat{E}_{n=0} = -2\mu(1 + \frac{A_3}{\sqrt{2A_4}}) \quad (26)$$

Using the physically acceptable definition of  $A_1$  in Eq. (20), the above equation can be rearranged as

$$\hat{E}_{n=0} = -\frac{2A_1}{|E_{n=0}|^{1/2}} \quad (27)$$

where  $E_{n=0}$  has been described in Eq. (19). This brief discussion shows explicitly the relation between the two singular potentials in higher dimensions and verifies Eq. (11).

### 3.2 Exactly solvable case

Kasap [27] and his co-workers used supersymmetric quantum mechanics to find exact results for the special cases of the singular potentials of (1) and (2), more precisely the solutions of the Kratzer and pseudoharmonic potentials in three dimensions. Their results can be easily generalized to  $N$ -dimensions by the substitution  $\ell \rightarrow \Lambda = (M - 3)/2$  as indicated in section II. This extension to arbitrary dimensions helps us in constructing the map between these two distinct systems.

The study of anharmonic oscillators has raised a considerable amount of interest because of its various applications especially in molecular physics. The Morse potential is commonly used for anharmonic oscillator. However, its wave function does not vanish at the origin, but those for Mie-type and pseudoharmonic potentials do. The Mie-type potential possesses the general features of the true interaction energy, inter-atomic and inter-molecular, and dynamical properties of solids [28]. On the other hand, the pseudoharmonic potential may be used for the energy spectrum of linear and non-linear systems [20]. The Mie-type and pseudo-harmonic potentials are two special kinds of analytically solvable singular-power potentials as they have the property of shape-invariance.

Starting with the general form of the Mie-type potential

$$V(r) = D_0 \left[ \frac{p}{q-p} \left( \frac{\sigma}{r} \right)^q - \frac{q}{q-p} \left( \frac{\sigma}{r} \right)^p \right] \quad (28)$$

where  $D_0$  is the interaction energy between two atoms in a molecular system at  $r = \sigma$ , and  $q > p$  is always satisfied. If we take  $q = 2p$  and  $p = 1$ , we arrive at a special case of the potential in Eq. (28), which is exactly solvable

$$V(r) = \frac{A}{r^2} - \frac{B}{r} \quad (29)$$

where  $A = D_0\sigma^2$  and  $B = 2D_0\sigma$ . The above potential, the so-called Kratzer potential, includes the terms which give the representation of both the steep repulsive branch and the long-range attraction. A single minimum occurs at  $r = \sigma$  where the energy is  $-D_0$ . Considerable interest has recently been shown in this potential as a model to describe inter-nucleon vibration [29] and, in applications this Mie type potential offers one of the most important exactly solvable models of atomic and molecular physics and quantum chemistry [30].

We set the superpotential for the Kratzer effective potential

$$W(r) = \frac{B/2}{\beta + (\beta^2 + C)^{1/2}} - \frac{\beta + (\beta^2 + C)^{1/2}}{r} \quad (30)$$

where

$$C = \frac{\Lambda(\Lambda + 1)}{2} + A, \quad \Lambda = \ell + \frac{1}{2}(N - 3), \quad \beta = \frac{1}{2\sqrt{2}} \quad (31)$$

and obtained the exact spectrum in  $N$ -dimensional space as

$$E_n = \left( \frac{B/2\beta}{2n + 1 + [(2\Lambda + 1)^2 + A/\beta^2]^{1/2}} \right)^2, \quad n = 0, 1, 2, \dots \quad (32)$$

and from Eq. (13) the exact unnormalized ground state wavefunction can be expressed as

$$\Psi_{n=0}(r) = r^{1/2\{1+[(2\Lambda+1)^2+A/\beta^2]^{1/2}\}} \times \exp\left(-\frac{Br/4\beta^2}{1 + [(2\Lambda + 1)^2 + A/\beta^2]^{1/2}}\right) \quad (33)$$

The excited state wavefunctions can be easily determined from the usual approach in supersymmetric quantum mechanics [9] and the normalization coefficients for each quantum state wave function can be analytically worked out using the explicit recurrence relation given in a recent work [31].

As a second application, we consider the general form of the pseudoharmonic potential

$$\tilde{V}(r) = V_0\left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2 = \tilde{B}r^2 + \frac{\tilde{A}}{r^2} - 2V_0 \quad (34)$$

which can be used to calculate the vibrational energies of diatomic molecules with the equilibrium bond length  $r_0$  and force constant  $k = 8V_0/r_0^2$ , and set the corresponding superpotential as

$$W(r) = \sqrt{\tilde{B}}r - \frac{\beta + (\beta^2 + \tilde{C})^{1/2}}{r} \quad (35)$$

where  $\tilde{B} = V_0/r_0^2$ ,  $\tilde{C} = [\Lambda(\Lambda + 1) + 2\tilde{A}]/2$ ,  $\tilde{A} = V_0r_0^2$ . The exact full spectrum of the potential in arbitrary dimensions is

$$\tilde{E}_n = 2\beta\sqrt{\tilde{B}}\{4n + 2 + [(2\Lambda + 1)^2 + \tilde{A}/\beta^2]^{1/2}\} - 2V_0 \quad (36)$$

and the unnormalized exact ground state wave function is

$$\Psi_{n=0}(r) = r^{1/2\{1+[(2\Lambda+1)^2+\tilde{A}/\beta^2]^{1/2}\}} \times \exp\left(-\frac{\sqrt{\tilde{B}}r^2}{4\beta}\right) \quad (37)$$

Using the discussion in section 2, one can transform the Kratzer potential in Eq. (29) to its dual potential- shifted (by  $2V_0$ ) pseudoharmonic-like potential in Eq. (34) with some restrictions in potential parameters. In the light of Eqs. (9-11), the transformed potential reads

$$\hat{V}(\rho) = \rho^2 + \frac{4A}{\rho^2} \quad (38)$$

which is in the form of the Goldman-Krivchenkov potential. Here  $A(= D_0\sigma^2)$  is the Kratzer potential parameter and, considering Eqs. (34) through Eq. 36, constraints on the potential parameters are such that  $\tilde{B} = 1$  and  $\tilde{A} = 4A$ . In this case corresponding eigenvalues are

$$\begin{aligned}\hat{E}_{n'} &= \frac{2B}{|E_n|^{1/2}} = 4\beta\{1 + 2n' + [1 + 4\Lambda'(\Lambda' + 1) + \frac{A}{\beta^2}]\} , \\ \Lambda' &= L + \frac{1}{2}(N' - 3)\end{aligned}\quad (39)$$

where  $B(= 2D_0\sigma)$  and  $E_n$  are the coupling parameter and the eigenenergy values (Eq. 32), respectively, of the Kratzer potential.

The ensuing relationships among the dimensionalities and quantum numbers of the two distinct systems considered here in this section are :

$$N' = 2N - 2 - 2\lambda , \quad L = 2\ell + \lambda \quad n' = 2n - 2 + \lambda \quad (40)$$

Clearly, the mapping parameter  $\lambda$  must be an integer if  $n'$ ,  $L$ ,  $n$  and  $\ell$  are integers. It is worthwhile to discuss briefly the physics behind this transformation in the light of the comprehensive work of Kostecky *et al.* [22]. We note that it is a general feature of this map that the spectrum of the  $N$ -dimensional problem involving Kratzer potential is related to the half the spectrum of the  $N'$ -dimensional problem involving Goldman-Krivchenkov potential for any even integer  $N'$ . However, the quantities in Eq. (40) have parameter spaces that are further restricted by the properties chosen for the map. For instance, suppose we wish to map all states corresponding the  $N$ -dimensional Kratzer potential into those corresponding Goldman-Krivchenkov potential. Since on physical grounds we know that  $N' \geq 2$ ,  $n' \geq 0$ ,  $L \geq 0$ , we must impose  $N \geq 2 + \lambda$ ,  $n \geq 1 - \lambda/2$ ,  $\ell \geq -\lambda/2$ . This yields the bound  $-2\ell \leq \lambda \leq N - 2$ . Further requiring  $n \geq 1$ ,  $\ell \geq 0$  restricts the bound to  $0 \leq \lambda \leq N - 2$ . We conclude that all states of the  $N$ -dimensional Kratzer problem can be mapped into the appropriate Goldman-Krivchenkov problem, except for  $N = 1$ .

As an example, consider the three-dimensional Kratzer problem. Assuming we wish to map all its states into those of its dual-the Goldman-Krivchenkov potential, we must impose  $0 \leq \lambda \leq 1$ . First, take  $\lambda = 0$ . Then, the  $s$ -orbitals in Kratzer potential ( $n \geq 1, \ell = 0$ ) are related to the ( $n' = 2n - 2 \geq 0, L = 0$ ) states of the four-dimensional Goldman-Krivchenkov problem. Similarly, the  $p$ -states ( $n \geq 2, \ell = 1$ ) correspond to the ( $n' = 2n - 2 \geq 0, L = 0$ ) same problem. Next, suppose  $\lambda = 1$ . The states corresponding the potential in Eq. (29) are then mapped into the odd-integer states of the two-dimensional oscillator problem of Eq. (38). The  $s$ -orbitals of Kratzer potential ( $n \geq 1, \ell = 0$ ) map into the ( $n' = 2n - 1 \geq 1, L = 1$ ) anharmonic oscillator states corresponding Goldman-Krivchenkov potential, while the Kratzer  $p$ -orbitals ( $n \geq 2, \ell = 1$ ) map into the ( $n' = 2n - 1 \geq 3, L = 1$ ) oscillator states of Goldman-Krivchenkov problem. As a rule, in both cases ( $\lambda = 0, 1$ ), the lowest-lying states of Goldman-Krivchenkov potential are excluded, one by one, with each higher value of  $\ell$ .

As a final remark, a student of introductory quantum mechanics often learns that the Schrödinger equation is exactly solvable (for all angular momenta) for two central potentials in Eqs. (29) and (38), and for also their special cases ( $A = 0$ ) the Coulomb and harmonic oscillator problems. Less frequently, the student made aware of the relation between these two problems, which are linked by a simple change of the independent variable discussed in detail through the paper. Under this transformation, energies and coupling constants trade places, and orbital angular momenta are rescaled. Thus, we have in this section shown that there is really only one quantum mechanical problem, not two involving the Kratzer and Goldman-Krivchenkov potentials, which can be exactly solved for all orbital angular momenta.



## 4 Conclusion

The main aim of this work has been to establish a very general connection between a class of singular potentials in higher dimensional space through the application of a suitable transformation. Although much work had been done in the literature on similar problems, an investigation as the one we have discussed in this paper was missing to our knowledge. In addition, it is shown that the supersymmetric quantum mechanics yields exact solutions for a single state only for the quasi-exactly solvable potentials such as the ones given in Eqs. (1) and (2) with some restrictions on the potential parameters in  $N$ -dimensional space, unlike the shape invariant exactly solvable potentials. We have also shown how to obtain exact solutions to such problems in any dimension by applying an adequate transformation to previously known three-dimensional results. This simple and intuitive method discussed through this paper is easy to be generalized. The application of this method to other potentials involving non-central ones are in progress.

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